THE HOROSPHERICAL GEOMETRY OF SURFACES IN HYPERBOLIC 4-SPACE

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ABSTRACT

We study some geometrical properties associated to the contacts of surfaces with hyperhorospheres in $H^4_+(-1)$. We introduce the concepts of osculating hyperhorospheres, horobinormals, horoasymptotic directions and horospherical points and provide conditions ensuring their existence. We show that totally semiumbilical surfaces have orthogonal horoasymptotic directions.

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1. Introduction

A hypersurface given by the intersection of the hyperbolic n-space $H^n_+(-1)$ with a spacelike, timelike or lightlike hyperplane of $\mathbb{R}^{n+1}_1$ is respectively called **hypersphere, equidistant hyperplane** or **hyperhorosphere**. The last have the property of inheriting a euclidean geometry as submanifolds of the hyperbolic space. A great deal of the properties of submanifolds of euclidean spaces can be studied by analyzing their contacts with invariant subsets, such as hyperplanes or hyperspheres, of the ambient space as can be appreciated in the classical references [6], [14]. A modern treatment based on recently developed techniques of singularity theory can be found in [8], [9] or [10]. In a similar way, the study of the contacts of a submanifold $M \subset H^n_+(-1)$ with the hyperhorospheres leads to the *horospherical geometry* of M. An introduction to this for hypersurfaces in $H^n_+(-1)$ has been given in [3]. On the other hand, the geometry associated to the contacts of surfaces with lightlike hyperplanes in $\mathbb{R}^4_1$ has been described through the analysis of the singularities of the **lightcone height functions family** defined in [4]. These are closely related to those of the lightcone Gauss map. The contacts that concern us here, namely those of a surface $M = g(U) \subset H^4_+(-1)$ with hyperhorospheres, can be similarly described by means of the **lightcone height functions family** $\mathcal{H}$: $M \times S^3_+ \to \mathbb{R}$. This setting allows us to show that M may have a stronger contact with certain hyperhorospheres at some points. We call them **osculating hyperhorospheres.** They play a role which is equivalent to that of the osculating hyperplanes of surfaces immersed in euclidean 4-space ([8]). We notice that there is an essential difference: whereas in the euclidean case the maximum number of osculating hyperplanes at a point of a generic surface is 2, we shall show that in our case there may be up to four osculating hyperhorospheres (Proposition 4.4). We define the **horospherical points** as those at which some osculating hyperhorosphere has a contact of corank 2 with the surface. They are analogous, in the horospherical geometry, to the inflection points of the euclidean geometry. We show in Proposition 4.6 that the curvature ellipse at these points is degenerate (i.e., they are semiumbilic, see [5]). We characterize them as critical points of certain direction fields on M that we call **horoasymptotic**. We obtain some conditions that guarantee: (a) the global existence of such fields on the surface (Theorem 5.2), and (b) the existence of horospherical points (Corollary 5.3). This leads us to put forward the following horospherical version of Carathéodory's conjecture:

Any 2-sphere immersed as an everywhere horohyperbolic surface in hyperbolic 4-space has at least 2 horospherical points.

Finally, we show that, provided the horoasymptotic fields are globally defined on M, the total semiumbilicity implies the orthogonality of their integral lines (Theorem 5.4).

2. Basic concepts and notations

We consider the $(n+1)$-dimensional Minkowski space $(\mathbb{R}^{n+1}, \langle , \rangle)$, with the pseudo-scalar product given by

$$\langle (x_1, x_2, \dots, x_{n+1}), (y_1, y_2, \dots, y_{n+1}) \rangle = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}.$$

We shall denote this space by $\mathbb{R}^{n+1}_1$.

We say that a vector $\mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_1 \backslash \{0\}$ is **spacelike, timelike** or **lightlike** provided $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $= 0$ or < 0, respectively. The norm or *length* of a vector $\mathbf{x} \in \mathbb{R}^4_1$ is defined by $||\mathbf{x}|| = (|\langle \mathbf{x}, \mathbf{x} \rangle|)^{1/2}$.

Given a vector $\mathbf{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define a **hyperplane with pseudonormal** $\mathbf{v}$ as

$$P_{(\mathbf{v},c)} = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 | \langle \mathbf{x}, \mathbf{v} \rangle = c\}.$$

This hyperplane is said to be **spacelike, timelike** or **lightlike** according as $\mathbf{v}$ is timelike, spacelike or lightlike.

We define the **hyperbolic n-space** by

$$H^n_+(-1) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 | \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_1 \geq 1\}.$$

Any non-empty hypersurface of $H^n_+(-1)$ determined by the intersection of $H^n_+(-1)$ with either a spacelike, a timelike or a lightlike hyperplane is respectively called a **hypersphere, equidistant hyperplane** or **hyperhorosphere**.

Another relevant set, known as the **n-dimensional lightcone with vertex a in** $\mathbb{R}^{n+1}_1$, is the following:

$$LC_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 | \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\}.$$

The subset

$$S^{n-1}_+(\mathbf{a}) = \{\mathbf{x} = (x_1, x_2, \dots, x_{n+1}) | \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0, x_1 = a_1 + 1\}$$

is the **lightcone $(n-1)$-sphere** centered at $\mathbf{a} = (a_1, a_2, \dots, a_{n+1})$. It will be denoted as S^{n-1}_+ when centered at the origin.

Suppose that M is a surface immersed in $\mathbb{R}^{n+1}_1$. We say that M is a **spacelike surface** if the tangent plane $T_{\mathbf{x}}M$ is spacelike (i.e., consists of spacelike vectors) and thus a euclidean plane $(T_{\mathbf{x}}M, \langle , \rangle)$ for every point $\mathbf{x} \in M$. In this case, the normal space $N_{\mathbf{x}}M$ is a Lorentz $(n-1)$-space $((N_{\mathbf{x}}M, \langle , \rangle)$.

3. Second fundamental form and curvature ellipses

Given a smooth oriented surface M immersed in $\mathbb{R}^5_1$, we denote respectively by $\mathcal{X}(M)$ and $\mathcal{N}(M)$ the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M. Consider the second fundamental map,

$$\alpha\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{N}(M),\ \alpha(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y,$$

where $\bar{\nabla}$ denotes the pseudo-riemannian connection of $\mathbb{R}^5_1$, and $\bar{X}$ and $\bar{Y}$ are local extensions of the tangent vector fields X and Y on M. This map is well defined, symmetric and bilinear. Given any normal field $\nu \in \mathcal{N}(M)\backslash\{0\}$, we have for each $\mathbf{x} \in M$ a function

$$\begin{aligned} \alpha_\nu\colon T_{\mathbf{x}}M \times T_{\mathbf{x}}M &\longrightarrow \mathbb{R} \\ (\mathbf{v}, \mathbf{w}) \qquad &\longmapsto \langle \alpha(\mathbf{v}, \mathbf{w}), \nu(\mathbf{x}) \rangle, \end{aligned}$$

which is also symmetric and bilinear. The second fundamental form of M at $\mathbf{x}$ is the associated quadratic form,

$$\begin{aligned} II_\nu : T_{\mathbf{x}}M &\longrightarrow \mathbb{R} \\ \mathbf{v} \quad &\longmapsto \alpha_\nu(\mathbf{v}, \mathbf{v}). \end{aligned}$$

Suppose that M is locally defined at $\mathbf{x}$ by $g\colon \mathbb{R}^2 \to \mathbb{R}^5_1$ such that $g(0,0) = \mathbf{x}$. Choose local isothermic coordinates $\{x, y\}$ on M and a pseudo-orthonormal frame, $\{e_1, e_2, e_3, e_4, e_5\}$ in a neighborhood of $\mathbf{x} = g(0,0) \in M$, such that $\{e_1, e_2, e_3\}$ is a normal frame and $\{e_4, e_5\}$ a tangent frame, with $\langle e_1, e_1 \rangle = -1$ and $\langle e_i, e_i \rangle = 1$, $i = 2, \ldots, 5$. Then the matrix of the bilinear form α_{e_i} is given by

$$\alpha_{e_i}(\mathbf{x}) = \begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix},$$

where if $ds^2 = E(dx^2 + dy^2)$ is the first fundamental form, we have

$$a_i = \frac{1}{E}\left\langle \frac{\partial^2 g}{\partial x^2}(0,0), e_i \right\rangle, \quad b_i = \frac{1}{E}\left\langle \frac{\partial^2 g}{\partial x \partial y}(0,0), e_i \right\rangle, \quad c_i = \frac{1}{E}\left\langle \frac{\partial^2 g}{\partial y^2}(0,0), e_i \right\rangle,$$

for $i = 1, 2, 3$.

Given $\mathbf{x} \in M$, consider the linear map induced by the second fundamental form on M, $A_{\mathbf{x}} : N_{\mathbf{x}}M \to Q^2$, where Q^2 is the space of quadratic forms in two variables. That is, $A_{\mathbf{x}}(\mathbf{v}) = II_{\mathbf{v}}, \forall \mathbf{v} \in N_{\mathbf{x}}M$.

We have, in the above coordinates, that if $\mathbf{v} = v_1e_1 + v_2e_2 + v_3e_3$, then

$$A_{\mathbf{x}}(\mathbf{v}) = \frac{1}{E}(v_1\langle d^2g(0,0), e_1\rangle + v_2\langle d^2g(0,0), e_2\rangle + v_3\langle d^2g(0,0), e_3\rangle).$$

And thus the matrix of $A_{\mathbf{x}}$ is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

We say that a point $\mathbf{x} \in M$ is of type M_i, $i = 3, 2, 1, 0$ provided $\operatorname{rank} A_{\mathbf{x}} = i$. It was shown in ([9], Propositions 2 and 3) that *for generic surfaces in euclidean 5-space the M_3-points fill an open and dense submanifold, whereas the M_2-points form closed regular curves and the M_1-points and M_0-points can be avoided.* We observe that the arguments of [9] can be easily adapted to our case, so we can conclude that *the same assertions hold for generic spacelike surfaces immersed in 5-dimensional Minkowski space.*

Given $\mathbf{x} \in M$, consider the unit circle in $T_{\mathbf{x}}M$ parametrized by the angle $\theta \in [0, 2\pi]$. Denote by γ_θ the spacelike curve obtained by intersecting M with the timelike hyperplane defined by the direct sum of the normal subspace $N_{\mathbf{x}}M$ and the straight line in the tangent direction represented by θ. The curvature vector $\eta(\theta)$ of γ_θ in $\mathbf{x}$ lies in the timelike hyperplane $N_{\mathbf{x}}M$. Varying θ from 0 to 2π, the vector $\eta(\theta)$ describes an ellipse in $N_{\mathbf{x}}M$, called the **curvature ellipse** of M at $\mathbf{x}$. This ellipse is the image of the affine map (the case $n = 3$ has been described in [5] and the case $n \geq 4$ is a straightforward generalization)

$$\eta\colon S^1 \subset T_{\mathbf{x}}M \longrightarrow N_{\mathbf{x}}M$$

given by

$$\theta \longmapsto \eta(\theta) = \sum_{i=1}^{3} [\cos\theta \quad \sin\theta]. \begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot e_i,$$

that is,

$$\eta(\theta) = D_{\mathbf{x}} + B_{\mathbf{x}} \cos 2\theta + C_{\mathbf{x}} \sin 2\theta,$$

where

$$D_{\mathbf{x}} = \frac{1}{2}(a_1 + c_1)e_1 - \frac{1}{2}\sum_{i=2}^{3}(a_i + c_i) \cdot e_i,$$

$$B_{\mathbf{x}} = \frac{1}{2}(a_1 - c_1)e_1 - \frac{1}{2}\sum_{i=2}^{3}(a_i - c_i) \cdot e_i,$$

$$C_{\mathbf{x}} = b_1e_1 - \sum_{i=2}^{3} b_i \cdot e_i.$$

LEMMA 3.1: *Given a spacelike surface $M \subset \mathbb{R}^5_1$, the subspace* $\operatorname{Ker} A_{\mathbf{x}}$ *determined by the kernel of $A_{\mathbf{x}}$ in $N_{\mathbf{x}}M$ is pseudo-orthogonal to the vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$ that define the curvature ellipse.*

Proof: We can assume that $\mathbf{x} \in M_i$, $i < 3$, otherwise $\operatorname{Ker} A_{\mathbf{x}} = \{0\}$ and the result is trivial. Given $\mathbf{v} = v_1 e_1 + v_2 e_2 + v_3 e_3 \in \operatorname{Ker} A_{\mathbf{x}}$, we have that $a_1 v_1 + a_2 v_2 + a_3 v_3 = b_1 v_1 + b_2 v_2 + b_3 v_3 = c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. Then $2\langle \mathbf{v}, B_{\mathbf{x}} \rangle = -v_1(a_1 - c_1) - v_2(a_2 - c_2) - v_3(a_3 - c_3) = 0$ and $\langle \mathbf{v}, C_{\mathbf{x}} \rangle = -v_1 b_1 - v_2 b_2 - v_3 b_3 = 0$. ∎

The curvature ellipse at $\mathbf{x}$ is contained in the Lorentz 3-space $N_{\mathbf{x}}M$ and may degenerate (to a segment, or even to a point) at certain points of $\mathbf{x} \in M$. These are called **semiumbilics**. A semiumbilic point $\mathbf{x}$ is said to be **spacelike, timelike** or **lightlike** provided the curvature segment defines respectively a spacelike, timelike or lightlike direction in $N_{\mathbf{x}}M$. The points at which the curvature ellipse becomes a point are known as **umbilics**. It is a straightforward exercise to verify that any semiumbilic point is of type $M_i, i < 3$. We notice that although M_1-points are either semiumbilic or umbilic, not every point of type M_2 needs to be a semiumbilic. Moreover, it was shown in [11] that the semiumbilics of generically immersed surfaces in euclidean 5-space are isolated points (lying on curves of M_2-points) and it is not difficult to see that similar arguments apply to the case of surfaces generically immersed in Minkowski 5-space. A surface all of whose points are semiumbilic is said to be **totally semiumbilical**. Some of the properties of totally semiumbilical surfaces in $\mathbb{R}^{n+1}_1$ are studied in [5]. In particular, for surfaces contained in hyperbolic 4-space we have

PROPOSITION 3.2: *Given a surface $M \subset H^4_+(-1)$, the curvature ellipse of M at a point $\mathbf{x} \in M$ is contained in an affine plane of $N_{\mathbf{x}}M$ parallel to $T_{\mathbf{x}}H^4_+(-1) \cap N_{\mathbf{x}}M$.*

Proof: Consider the position field $\rho(\mathbf{x}) = \mathbf{x}$ on M. In isothermic coordinates, $\{x, y\}$, over M, this normal field satisfies

$$\langle \mathbf{x}_{xx}, \rho \rangle = -\langle \mathbf{x}_x, \mathbf{x}_y \rangle = -E,$$
$$\langle \mathbf{x}_{xy}, \rho \rangle = -\langle \mathbf{x}_x, \mathbf{x}_y \rangle = -F = 0,$$
$$\langle \mathbf{x}_{yy}, \rho \rangle = -\langle \mathbf{x}_y, \mathbf{x}_y \rangle = -G = -E,$$

where E, F and G are the coefficients of the first fundamental form on M.

Now, if we express $\rho = \sum_{i=1}^{3} \rho_i e_i$ in terms of a pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ as above, we have

$$\langle B_{\mathbf{x}}, \rho \rangle = -\sum_{i=1}^{3} \rho_i (a_i - c_i) = -\langle \rho, \mathbf{x}_{xx} \rangle + \langle \rho, \mathbf{x}_{yy} \rangle = 0,$$

$\langle C_{\mathbf{x}}, \rho \rangle = -\sum_{i=1}^{3} \rho_i b_i = \langle \rho, \mathbf{x}_{xy} \rangle = 0.$

Therefore, the plane determined by the vectors $C_{\mathbf{x}}$ and $B_{\mathbf{x}}$ in $N_{\mathbf{x}}M$ is pseudo-orthogonal to ρ. Since this is pseudo-orthogonal to the hyperbolic space $H^4_+(-1)$, we have the required result. ∎

As a consequence of this, we see in the next section that the generic behavior of semiumbilic points of surfaces contained in $H^4_+(-1)$ differs from that of surfaces in $\mathbb{R}^5_1$.

The **shape operator** associated to a normal field ν is defined as

$$S_\nu : TM \to TM, \quad S_\nu(X) = -(\bar{\nabla}_{\bar{X}} \bar{\nu})^{\top},$$

where $\bar{\nu}$ is a local extension to $\mathbb{R}^5_1$ of the normal vector field ν at $\mathbf{x}$ and $()^{\top}$ means the tangent component. This operator is bilinear, self-adjoint and satisfies the following equation: $\langle S_\nu(X), Y \rangle = H_\nu(X,Y)$, $\forall X, Y \in \mathcal{X}(M)$. So we have that $II_\nu(X) = \langle S_\nu(X), X \rangle$.

We can find, for each $\mathbf{x} \in M$, an orthonormal basis of $T_{\mathbf{x}}M$ consisting of eigenvectors of S_ν, for which the restriction of the second fundamental form to the unit vectors $II_\nu|_{S^1}$ takes its maximal and minimal values. The corresponding eigenvalues k_1, k_2 are the *ν-principal curvatures.* A point $\mathbf{x}$ is said to be *ν-umbilic* if both ν-principal curvatures coincide at $\mathbf{x}$. Let $\mathcal{U}_\nu$ be the set of ν-umbilics in M. For any $\mathbf{x} \in M \backslash \mathcal{U}_\nu$ there are two ν-principal directions defined by the eigenvectors of S_ν. These are smooth integrable direction fields and their integrals define two families of orthogonal curves which are called the **ν-principal lines of curvature**. The two orthogonal foliations with the ν-umbilics as its singularities form the **ν-principal configuration** of M. We say that the surface M is **ν-umbilical** if each point of M is ν-umbilic. Some umbilicity properties of surfaces immersed in Minkowski spaces have been studied in [5]. In particular, it was shown (Proposition 5.1) that a point $\mathbf{x}$ of a surface $M \subset \mathbb{R}^5_1$ is ν-umbilic for some normal field ν if and only if $\nu(\mathbf{x})$ is pseudo-orthogonal to the vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$ that define the curvature ellipse at $\mathbf{x}$. Then for the particular case of a surface $M \subset H^4_+(-1)$, as a consequence of Proposition 3.2, we have:

If $\rho(\mathbf{x}) = \mathbf{x}$ is the position (normal) field on M, then each point of M is ρ-umbilical.

On the other hand, we quote the following results obtained in [5]:

PROPOSITION 3.3: *A surface $M \subset H^4_+(-1)$ is totally semiumbilical if and only if M is umbilical with respect to some lightlike normal field.*

COROLLARY 3.4: *A surface $M \subset H^4_+(-1)$ lies in a hyperhorosphere if and only if it is umbilical with respect to some lightlike normal field ν with constant zero curvature.*

4. Contacts with lightlike hyperplanes and hyperhorospheres

Suppose that M and N are submanifolds of $\mathbb{R}^{n+1}$ locally defined by $M = g(\mathbb{R}^m)$ and $N = f^{-1}(0)$, where $g\colon \mathbb{R}^m \to \mathbb{R}^{n+1}$ is an embedding and $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^q$ is a submersion. We can measure the contact of M and N at a common point $p \in M \cap N$ by analyzing the singularities of the map $f \circ g\colon \mathbb{R}^m \to \mathbb{R}^q$ (contact map at p). In fact, if M and N are submanifolds of a manifold Z and M' and N' are submanifolds of Z', we say that M and N have the same contact at a point p as M' and N' at p' provided there exists a diffeomorphism germ $\phi\colon (Z,p) \to (Z',p')$ taking M to M' and N to N'. In such a case we write $K(M,N) = K(M',N')$. J. A. Montaldi proved [10] that this holds if and only if their respective contact map-germs at p and p' are $\mathcal{K}$-equivalent. Here, we say that two map-germs $f_i\colon (\mathbb{R}^m, x_i) \to (\mathbb{R}^p, y_i)$, $i = 1,2$ are **$\mathcal{K}$-equivalent** (denoted $\mathcal{K}(f_1) = \mathcal{K}(f_2)$) if there is a diffeomorphism-germ (contact-equivalence), $\Phi\colon (\mathbb{R}^m \times \mathbb{R}^p, (x_1, y_1)) \to (\mathbb{R}^m \times \mathbb{R}^p, (x_2, y_2))$ of the form $\Phi(x,t) = (h(x), \theta(x,t))$, such that $\Phi(x, y_1) = (h(x), y_2)$ and $\Phi(x, f_1(x)) = (h(x), f_2(h(x)))$. We refer to [2] or [7] for the definition and details on $\mathcal{K}$-equivalence.

Therefore, to study the contact of a spacelike surface locally given as $M = g(\mathbb{R}^2) \subset \mathbb{R}^5_1$ with some hyperplane whose pseudonormal vector is $\mathbf{v}$, $P_{(\mathbf{v},c)}$ at a given point $\mathbf{x} = g(u) \in M$, the map f has to be chosen in such a way that $P_{(\mathbf{v},c)} = f^{-1}(0)$. That is,

$$f(x_1, \ldots, x_5) = -x_1 \cdot v_1 + \cdots + x_5 \cdot v_5 - c,$$

where $\mathbf{v} = (v_1, \ldots, v_5)$ and $\mathbf{x} = (x_1, \ldots, x_5)$.

To analyze all the possible contacts of the submanifold $M = g(\mathbb{R}^2)$ with the lightlike hyperplanes of $\mathbb{R}^5$, we must describe the singularities of the **lightcone height functions family**

$$\begin{aligned} \mathcal{H}\colon \mathbb{R}^2 \times S^3_+ &\longrightarrow \mathbb{R} \\ (u, \mathbf{v}) &\longmapsto \langle g(u), \mathbf{v} \rangle. \end{aligned}$$

We shall denote by $h_{\mathbf{v}}$ the function obtained when fixing the parameter $\mathbf{v}$. Clearly, u is a singular point of $h_{\mathbf{v}}$ if and only if $\mathbf{v} \in N_{\mathbf{X}(u)}M$.

Suppose now that M lies in $H^4_+(-1)$. Given $\mathbf{v} \in S^3_+$ and $\mathbf{x} \in M$, we denote by $\Omega(\mathbf{v}, \mathbf{x})$ the hyperhorosphere of $H^4_+(-1)$ determined by the lightlike hyperplane

$P_{\mathbf{v},c}$ with pseudo-normal $\mathbf{v}$ that passes through the point $\mathbf{x} = g(u)$ (i.e., $\langle \mathbf{x}, \mathbf{v}\rangle = c$). We have that $\Omega(\mathbf{v}, \mathbf{x})$ is tangent to M at $\mathbf{x}$ if and only if u is a singular point of $h_{\mathbf{v}}$. Furthermore,

LEMMA 4.1: *Given a surface $M = g(\mathbb{R}^2) \subset H^4_+(-1)$ and $\mathbf{x} \in M$, the contact map-germs of the pairs $(M, \Omega(\mathbf{v}, \mathbf{x}))$ and $(M, P_{(\mathbf{v},c)})$ at $\mathbf{x}$ coincide.*

Proof: We have that $P_{(\mathbf{v},c)} = h^{-1}_{(\mathbf{v},c)}(0)$, where $h_{(\mathbf{v},c)}\colon \mathbb{R}^5_1 \to \mathbb{R}$ is given by $h_{(\mathbf{v},c)}(\mathbf{p}) = \langle \mathbf{v}, \mathbf{p}\rangle - c$. So if we represent by $i : H^4_+(-1) \to \mathbb{R}^5_1$ the canonical inclusion, we have that the contact map for $P_{(\mathbf{v},c)}$ and M is given by $h_{(\mathbf{v},c)} \circ i \circ g$. On the other hand, if we denote $\bar{h}_{(\mathbf{v},c)} = h_{(\mathbf{v},c)}|_{H^4_+(-1)}$, we have that $\bar{h}^{-1}_{(\mathbf{v},c)}(0) = P_{(\mathbf{v},c)} \cap H^4_+(-1) = \Omega(\mathbf{v}, \mathbf{x})$, and hence $\bar{h}_{(\mathbf{v},c)} \circ g$ is the contact map for M and $\Omega(\mathbf{v}, \mathbf{x})$. But, clearly, $\bar{h}_{(\mathbf{v},c)} \circ g = h_{(\mathbf{v},c)} \circ i \circ g$. ∎

Given a singular point u of the function $h_{\mathbf{v}}$, we say that $\mathbf{v}$ is a **horobinormal** direction for M at $\mathbf{x} = g(u)$ if the hessian matrix $Hessh_{\mathbf{v}}(u)$ defines a degenerate quadratic form. In this case, we have that $\Omega(\mathbf{v}, \mathbf{x})$ has higher order contact with M at $\mathbf{x}$ and we call it an **osculating hyperhorosphere**. A normal field ν defined on some open subset V of M and such that $\nu(\mathbf{x})$ is a horobinormal direction at $\mathbf{x}, \forall \mathbf{x} \in V$, is called a **horobinormal field on** V.

Given $\mathbf{x} \in M$, consider the linear map $A_{\mathbf{x}}\colon N_{\mathbf{x}}M \to Q^2$ and denote by C the cone of degenerate quadratic forms in Q^2. Clearly, $\mathbf{v} \in N_{\mathbf{x}}M$ determines a horobinormal if and only if $\mathbf{v} \in A^{-1}_{\mathbf{x}}(C) \cap LC_{\mathbf{x}}$.

A particular feature of the spacelike surfaces contained in hyperbolic 4-space is the following:

LEMMA 4.2: *The points of type M_2 of a surface $M \subset H^4_+(-1)$ are all semiumbilic.*

Proof: Take a point $\mathbf{x} \in M$ of type M_2 and suppose that M is locally defined at $\mathbf{x}$ by an embedding $g\colon \mathbb{R}^2 \to H^4_+(-1)$, such that $\mathbf{x} = g(0,0)$. Then the height function $h_{\mathbf{x}}(u) = \langle g(u), \mathbf{x}\rangle - \langle g(0,0), \mathbf{x}\rangle = \langle g(u), \mathbf{x}\rangle + 1$ describes the contact of M at $\mathbf{x}$ with the hyperplane, $P_{\mathbf{x}}$, that is pseudo-orthogonal to the position vector $\rho(\mathbf{x})$ and passes through $\mathbf{x}$. By taking g in the Monge form, $g(u) = (u, g_1(u), g_2(u), g_3(u))$, it is not difficult to verify that the hessian matrix of $h_{\mathbf{x}}$ at $(0,0)$ coincides with that of $A_{\mathbf{x}}(\mathbf{x})$. If we assume now that $\mathbf{x}$ is not a semiumbilic point, it follows from Lemma 3.1, together with Proposition 3.2, that $\operatorname{Ker} A_{\mathbf{x}}$ is spanned by the position vector $\mathbf{x}$ and hence $Hessh_{\mathbf{x}}(0,0)$ is the null matrix. This means that $(0,0)$ is a non-stable singularity of $h_{\mathbf{x}}$, which implies that the extension of this function to $H^4_+(-1)$ also has a non-stable singularity at $\mathbf{x} \in H^4_+(-1)$. But it can be seen that the contacts of $H^4_+(-1)$ with all its tangent hyperplanes are non-degenerate in the sense that they lead to a stable height function. So we arrive at a contradiction. ∎

This leads to the following result, concerning the distribution of semiumbilic points over surfaces generically immersed in $H^4_+(-1)$, which supposes an interesting difference with respect to the generic behaviour of surfaces immersed in Minkowski 5-space.

PROPOSITION 4.3: *Given a surface M generically immersed in $H^4_+(-1)$, the points of type M_3 fill an open and dense submanifold and the semiumbilic points are all of type M_2 and define a closed curves embedded in M. Umbilic points do not appear on these surfaces.*

Proof: Let $\Delta(\mathbf{x}) = \det A_{\mathbf{x}}$. It is clear that $\Delta^{-1}(0) = M - M_3$. Since Δ is a continuous function on M, we have that M_3 must be an open region in M. The condition that $\mathbf{x} \in M_2$ implies, by Lemma 4.2, that $\mathbf{x}$ is semiumbilic. But this amounts to saying that the normal vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$ are linearly dependent. Since by Proposition 3.2 we know that $B_{\mathbf{x}}, C_{\mathbf{x}} \in T_{\mathbf{x}}H^4_+(-1)$, it follows that the position vector $\mathbf{x}$ must be pseudo-orthogonal to both $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$. We can consider that M is locally given as an embedding $g\colon \mathbb{R}^2 \to H^4_+(-1) \subset \mathbb{R}^5_1$ in the Monge form at $\mathbf{x}$. Then we take a pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ for M in a neighborhood of the point $\mathbf{x}$ in such a way that e_1 is the position vector field, e_2 and e_3 are normal vector fields while e_4 and e_5 generate the tangent planes. In these coordinates, we can write

$$B_{\mathbf{x}} = -\frac{1}{2}(a_2 - c_2) \cdot e_2 - \frac{1}{2}(a_3 - c_3) \cdot e_3 \quad \text{and} \quad C_{\mathbf{x}} = -b_2 \cdot e_2 - b_3 \cdot e_3.$$

Then the linear dependence of these two vectors is given by the requirement

$$\frac{a_2 - c_2}{b_2} = \frac{a_3 - c_3}{b_3},$$

which defines a 1-codimensional algebraic variety of the jet space $J^2(\mathbb{R}^2, \mathbb{R}^5)$. It follows now from the Thom Transversality Theorem ([2]) that the 2-jet extension, $j^2 g\colon \mathbb{R}^2 \to J^2(\mathbb{R}^2, \mathbb{R}^5)$, meets this submanifold transversally. Therefore, the considered points determine an algebraic subset of codimension 1 in M.

On the other hand, the condition that $\mathbf{x} \in M_1 \cup M_0$ is equivalent to asking that $\operatorname{rank} A_{\mathbf{x}} \leq 1$. This means that the vector $H_{\mathbf{x}}$ must also be parallel to both $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$. This provides at least three independent quadratic equations in $J^2(\mathbb{R}^2, \mathbb{R}^5)$ and, by using again the Thom Transversality Theorem, we can conclude that, for a generic g, j^2g does not meet the corresponding algebraic variety, and thus $M_1 \cup M_0 = \emptyset$.

So $\Delta^{-1}(0) = M_2$ is completely made of semiumbilic points. We see now that they form embedded curves. In fact, let

$$g\colon \mathbb{R}^2, 0 \longrightarrow \mathbb{R}^5_1$$
$$(u,v) \longmapsto (u, v, g_1(u,v), g_2(u,v), g_3(u,v))$$

be the local representation of M in the Monge form at $\mathbf{x} \in M_2$. In these coordinates,

$$\begin{aligned}\Delta(\mathbf{x}) =& g_{1uu}g_{2uv}g_{3vv} - g_{1uv}g_{2uu}g_{3vv} - g_{1uu}g_{2vv}g_{3uv} \\ &+ g_{1vv}g_{2uu}g_{3uv} + g_{1uv}f_{2vv}f_{3uu} - f_{1vv}f_{2uv}f_{3uu}.\end{aligned}$$

It follows from this expression that, under appropriate transversality conditions on the 3-jet of g, the set $\Delta = 0$ represents a curve, possibly with isolated singular points determined by the vanishing of the derivatives of the function Δ. We observe that the pseudo-orthogonality property of the frame $\{e_1, e_2, e_3, e_4, e_5\}$ is irrelevant for our study. For a change of basis in $N_{\mathbf{x}}M$ preserves the relative position of $Im(A_{\mathbf{x}})$ with the cone C in Q^2, and thus preserves the sets M_3 and M_2. So we can take $\{e_1, e_2, e_3\}$ such that e_1 generates $\mathrm{Ker}(A_{\mathbf{x}})$.

If $p \in M_2$, we have three possibilities:

(i) $Im(A_{\mathbf{x}}) \cap C$ is a couple of lines,
(ii) $Im(A_{\mathbf{x}}) \cap C$ is a line, and
(iii) $Im(A_{\mathbf{x}}) \cap C$ is just the origin.

In case (i) we can choose $\{e_2, e_3\}$ as the two (degenerate) directions lying in $A^{-1}(C) \subset N_pM$. Furthermore, we can also make a change of coordinates in the source such that the two degenerate directions correspond to the quadratic forms u^2 and v^2 in C. Thus, g can be locally written as

$$g(u,v) = (u, v, u^2 + R_1(u,v), v^2 + R_2(u,v), R_3(u,v)),$$

where $R_i \in m^3$, i.e., all the derivatives of the R_i vanish up to order $3, i = 1, 2, 3$.

In case (ii), $Im(A_{\mathbf{x}})$ is tangent to C and we take e_3 as the generator of $A_{\mathbf{x}}^{-1}(Im(A_{\mathbf{x}}) \cap C)$. With additional changes of coordinates in the source, g can be written as

$$g(u,v) = (u, v, u^2 - v^2 + R_1(u,v), uv + R_2(u,v), R_3(u,v)).$$

Finally, in case (iii), all the quadratic forms $A_{\mathbf{x}}(\mathbf{v})$ are hyperbolic, and g can be written as

$$g(u,v) = (u, v, u^2 + R_1(u,v), uv + R_2(u,v), R_3(u,v)).$$

In each of the above cases it is a simple (but tedious) calculation to verify that, under generic conditions on the 3-jet of g at $(0,0)$, the derivatives of the function Δ do not vanish at $\mathbf{x}$ and thus it is a regular point of $\Delta^{-1}(0)$. ∎

We analyze next all the possibilities that we may have for the sets $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}}$ at different points $\mathbf{x} \in M_i$, $i = 3, 2, 1, 0$:

(a) If $\mathbf{x} \in M_3$, then $A_{\mathbf{x}}^{-1}(C)$ is a non-degenerate cone. The intersection $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}}$ depends on the relative position of both cones and may consist of either four, three, two, one or zero lines in $N_{\mathbf{x}}M$. We remark that it may also happen that both cones $A_{\mathbf{x}}^{-1}(C)$ and $LC_{\mathbf{x}}$ coincide, but this is an extremely degenerate phenomena that can be generically avoided, so we shall not consider it here.

(b) If $\mathbf{x} \in M_2$, as we have seen before, the plane $ImA_{\mathbf{x}}$ intersects the cone C in either (i) two lines, (ii) one line, or (iii) just the origin. In this case $A_{\mathbf{x}}^{-1}(C)$ is respectively made of (i) two planes with the common line $\operatorname{Ker} A_{\mathbf{x}}$, (ii) a plane containing the line $\operatorname{Ker} A_{\mathbf{x}}$, or (iii) just the line $\operatorname{Ker} A_{\mathbf{x}}$. Again, the intersection $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}}$ depends on the relative positions of both subsets and will consist of at most four lines in $N_{\mathbf{x}}M$.

(c) If $\mathbf{x} \in M_1$, then $ImA_{\mathbf{x}}$ is a line that may either lie on C, or intersect it just at the origin. Correspondingly, $A_{\mathbf{x}}^{-1}(C)$ will be the whole normal space $N_{\mathbf{x}}M$, or a plane. It follows that $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}}$ is the whole $LC_{\mathbf{x}}$ in the first case, or at most two lines in the second.

(d) Finally, if $\mathbf{x} \in M_0$, then $A_{\mathbf{x}}^{-1}(C)$ is the whole normal space and $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}} = LC_{\mathbf{x}}$.

Therefore, by taking into account that generic surfaces in $H_+^4(-1)$ are exclusively made of points of types M_3 and semiumbilics (M_2), we can state the following

PROPOSITION 4.4: *The number of osculating hyperhorospheres at any point of a surface M generically immersed in $H_+^4(-1)$ is at most four.*

A point $\mathbf{x} \in M$ of type M_3 or M_2 is said to be **horoelliptic** if the subset $A_{\mathbf{x}}^{-1}(C)$ intersects $LC_{\mathbf{x}}$ just at the origin. On the other hand, it is said to be **horohyperbolic** or **horoparabolic** according to whether they have transversal or non-transversal intersections off the origin. It is not difficult to verify that, generically, horoelliptic and horohyperbolic points determine open submanifolds of M separated by horoparabolic curves. For the particular case of a non-semiumbilic M_2-point x, we observe that x is necessarily horohyperbolic, having four horobinormals in the case (b,i), horoparabolic with two horobinormals in the case (b,ii), and horoelliptic with no horobinormals in the case (b,iii). On the other hand, if $x \in M_2$ is semiumbilic then both cases (b,i) and (b,ii) may correspond to either horohyperbolic, horoparabolic or horoelliptic points. Here we observe that horohyperbolic points may have either four or two horobinormal directions, due to the fact that the line $\operatorname{Ker} A_x$ does not need to lie inside the lightcone. Taking into account these considerations, we conclude

PROPOSITION 4.5: *If $M \subset H^4_+(-1)$ is exclusively made of horohyperbolic points, then it has either four or two globally defined horobinormal fields.*

As a consequence of the methods developed by Montaldi ([10]), it can be shown that, analogously to what happens in the case of surfaces generically immersed in Euclidean space, the rank of $Hessh_{\mathbf{v}}(u)$, for any horobinormal $\mathbf{v}$, is 1 at most points $\mathbf{x} = \mathbf{X}(u) \in M$. The points at which this rank is 0 are those at which the surface is better approached by some hyperhorosphere in all the tangent directions. These are analogous, in horospherical geometry terms, to the inflection points of surfaces in euclidean 4-space ([1]) and will be called **horospherical points.**

PROPOSITION 4.6: *The horospherical points of a surface M immersed in $H^4_+(-1)$ are either semiumbilics or umbilics. Moreover, every point of type M_1 or M_0 is a horospherical point.*

Proof: Suppose that M is given in the Monge form at a horospherical point $\mathbf{x} = g(0,0)$ and take a pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ in a neighborhood of $\mathbf{x}$ such that $\{e_4, e_5\}$ is a tangent frame and $\{e_1, e_2, e_3\}$ is a normal frame with $\langle e_1, e_1\rangle = -1$ as above. Then for any normal vector $\mathbf{v} \in N_{\mathbf{x}}M$, we can write $\mathbf{v} = v_1e_1 + v_2e_2 + v_3e_3$. We observe that the matrices $A_{\mathbf{x}}(\mathbf{v})$ and $Hessh_{\mathbf{v}}(0,0)$ coincide. Now, under the assumption that $\mathbf{x}$ is horospherical, we can choose a horobinormal vector $\mathbf{v} \in N_{\mathbf{x}}M \cap S^3_+(\mathbf{x})$ such that all the entries of the matrix $Hessh_{\mathbf{v}}(0,0)$ vanish. This implies that $II_{\mathbf{v}}$ also vanishes at $\mathbf{x}$. So we have that $-v_1a_1 + v_2a_2 + v_3a_3 = -v_1b_1 + v_2b_2 + v_3b_3 = -v_1c_1 + v_2c_2 + v_3c_3 = 0$, where the a_i, b_i, c_i, $i = 1, 2, 3$ are as in the previous section. But this means that the vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$, which determine the curvature ellipse at $\mathbf{x}$, are both pseudo-orthogonal to the lightlike direction $\mathbf{v}$. On the other hand, it follows from Proposition 3.2 that they are also pseudo-orthogonal to the timelike normal direction, $\mathbf{x}$, to $H^4_+(-1)$ at $\mathbf{x}$. Therefore, $\operatorname{rank}(B_{\mathbf{x}}, C_{\mathbf{x}}) \leq 1$. This implies that the curvature ellipse is degenerate at $\mathbf{x}$ and the first assertion is shown. To see the second, we observe that, given $\mathbf{x} \in M_1$, it follows from Lemma 3.1 that the plane $\operatorname{Ker} A_{\mathbf{x}}$ is pseudo-orthogonal to the direction determined by the (linearly dependent) vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$. But Proposition 3.2 implies that this direction is contained in the plane $T_{\mathbf{x}}H^4_+(-1)$. Therefore, we get that $KerA_{\mathbf{x}}$ must contain the line $\langle \mathbf{x} \rangle$ spanned by the position vector $\rho(\mathbf{x})$ at the point $\mathbf{x}$. So $KerA_{\mathbf{x}}$ cuts the lightcone at $\mathbf{x}$. This determines two horobinormals for which the hessian of the corresponding lightcone height function has rank 1, and hence $\mathbf{x}$ is a horospherical point. In case $\mathbf{x} \in M_0$, we have that all the horobinormals give rise to lightcone height functions whose hessian has rank 1 at $\mathbf{x}$. ∎

We remark, in particular, that the surfaces contained in a hyperhorosphere of $H^4_+(-1)$ are a special case of a totally semiumbilical surface, as can be concluded from Proposition 3.3 and Corollary 3.4.

5. Horoasymptotic directions

Given a surface M immersed in $H^4_+(-1)$, consider a local parametrization $\mathbf{X} : U \to H^4_+(-1)$ of M at a point $\mathbf{x}$. If $\mathbf{v} \in S^3_+(\mathbf{x})$ is a horobinormal of M at $\mathbf{x} = \mathbf{X}(u)$, we have that u is a degenerate singularity for the height function $h_{\mathbf{v}}$. Therefore, $\operatorname{Ker} Hess(h_{\mathbf{v}})(u) \neq \{0\}$. The non-zero directions lying in $\operatorname{Ker} Hess(h_{\mathbf{v}})(u)$ are called **horoasymptotic** directions at $\mathbf{x}$. We observe that these are the tangent directions at $\mathbf{x}$ along which the higher order contact of M and the hyperhorosphere $\Omega(\mathbf{v}, \mathbf{x})$ occurs. Clearly, any horobinormal field determines a (tangent) horoasymptotic field on the region of M over which it is defined. It follows from the definition of both horoasymptotic directions and horospherical points that the latter are the critical points of the horoasymptotic fields. We investigate next the possibilities of having some globally defined horoasymptotic field on M.

LEMMA 5.1:
- (i) *Two horobinormals $\mathbf{b}_1$ and $\mathbf{b}_2$ at a point $\mathbf{x} \in M$ of type M_3 cannot share horoasymptotic directions.*
- (ii) *Provided $\mathbf{x} \in M$ is of type M_i, $i < 3$, two horobinormals, $\mathbf{b}_1$ and $\mathbf{b}_2$, share a horoasymptotic direction if and only if they belong to some linear subspace contained in $A^{-1}_{\mathbf{x}}(C)$.*

Proof: (i) Suppose that θ is a common asymptotic direction for $\mathbf{b}_1$ and $\mathbf{b}_2$ at $\mathbf{x}$. In such a case we can choose coordinates on M at $\mathbf{x}$ in such a way that

$$Hess(h_{\mathbf{b}_1})(\mathbf{x}) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Hess(h_{\mathbf{b}_2})(\mathbf{x}) = \begin{bmatrix} \lambda_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

But then the normal direction $\mathbf{b} = \lambda_2\mathbf{b}_1 - \lambda_1\mathbf{b}_2$ has vanishing Hessian matrix at $\mathbf{x}$, which means that $\mathbf{x}$ is a horospherical point. By Proposition 4.3 this implies that $\mathbf{x}$ is a semiumbilic point and these are of type M_i, $i < 3$. So we have arrived at a contradiction.

(ii) A similar argument to the one used in part (i) tells us that common horoasymptotic directions for $\mathbf{b}_1$ and $\mathbf{b}_2$ at a point $\mathbf{x}$ lie in the kernel of any of their linear combinations. Therefore, $\mathbf{b}_1$ and $\mathbf{b}_2$ define a plane made of degenerate directions of $N_{\mathbf{x}}M$, in the sense that all of them are mapped by $A_{\mathbf{x}}$ into the cone C. Conversely, take any $\mathbf{b} \in A_{\mathbf{x}}^{-1}(C)$ that does not belong to $Ker A_{\mathbf{x}}$ (if all the degenerate directions lie in $\operatorname{Ker} A_{\mathbf{x}}$, we would have that all of them would share all the tangent directions in $T_{\mathbf{x}}M$ as horoasymptotic directions and the result would be trivially true). Then given any other $\mathbf{b}'$ lying in the same subspace than $\mathbf{b}$ in $A_{\mathbf{x}}^{-1}(C)$, we can always write $\mathbf{b}' = \lambda_1\mathbf{b} + \lambda_2\mathbf{b}''$, for some $\mathbf{b}'' \in \operatorname{Ker} A_{\mathbf{x}}$ and real numbers $\lambda_i, i = 1, 2$. It is not difficult to see that in this case $\mathbf{b}$ and $\mathbf{b}'$ share some horoasymptotic directions. ∎

THEOREM 5.2:

(i) *A generic surface $M \subset H_+^4(-1)$, all of whose points are horohyperbolic, has either two or four globally defined horoasymptotic fields that may eventually coincide pairwise over closed curves of semiumbilic points.*

(ii) *If M is totally semiumbilical with isolated horospherical points and such that all its points are horohyperbolic, then M has either one or two horoasymptotic fields globally defined that may eventually coincide over some closed subset.*

Proof: (i) In this case the horohyperbolicity ensures the existence of either two or four globally defined horobinormal fields which are not coincident at any point, otherwise this point would be horoparabolic. Then Lemma 5.1 (a) guarantees the existence of either two or four horoasymptotic fields respectively over the open surface M_3. Part (ii) of Lemma 5.1 tells us that the horoasymptotic directions coincide pairwise over the semiumbilic points. We recall that, according to Proposition 4.3, the semiumbilic points define closed curves on the generic surfaces.

(ii) Under this second asumption the horohyperbolicity condition ensures the existence of either two or four horobinormals at each point. Due to the absence of horoparabolic points, we can assert that these determine globally defined horobinormal fields on M, which in turn determine either one or two horoasymptotic fields. Again, we have that the horobinormal pairs may be coincident at points for which the image of $A_{\mathbf{x}}$ is tangent to the cone C. This implies that the two horoasymptotic directions will also be coincident at such points. We point out that in highly degenerate cases, this may happen over the clausure of some open region and even over the whole surface M. ∎

COROLLARY 5.3: *A generic surface in $H^4_+(-1)$ exclusively made of horohyperbolic points which is compact without boundary and has nonvanishing Euler number has horospherical points.*

Proof: This follows immediately by applying the Poincaré–Bendixon index formula to any of the horoasymptotic direction fields guaranteed by part (i) of Theorem 5.2. ∎

We observe that the same assertion is valid for surfaces satisfying the asumption (ii) of Theorem 5.2.

We recall that **Carathéodory's conjecture** asserts that a 2-sphere immersed in $\mathbb{R}^3$ has at least two umbilic points. We observe that the inverse of the stereographic projection takes umbilics of surfaces in 3-space to inflection points of their images in the S^3 considered as surfaces in euclidean 4-space. This naturally leads to the following **generalized Carathéodory's conjecture: 2-spheres convexly embedded in $\mathbb{R}^4$ have at least two inflection points**. Here, the convexity property is equivalent to the global existence of asymptotic directions (see [1] for a proof in the case of generic surfaces in euclidean 4-space). By analogy of this situation with the case considered here, we establish the following

CONJECTURE: *A 2-sphere immersed as an everywhere horohyperbolic surface in hyperbolic 4-space has at least two horospherical points.*

We finally find a relation between the semiumbilicity of a spacelike surface and the orthogonality of the horoasymptotic directions.

THEOREM 5.4: *Suppose that M is a surface in $H^4_+(-1)$ with two globally defined horoasymptotic fields and isolated horospherical points. If M is totally semiumbilical, then the horoasymptotic directions are mutually orthogonal everywhere except at the horospherical points.*

Proof: We observe first that M is totally semiumbilical if and only if there exist two normal fields ν_1, ν_2, locally defined and linearly independent at every non-umbilical point of M, such that M is ν_i-umbilical ([5], Theorem 5.6). We shall show now that this requirement on M is also equivalent to having a unique principal configuration. In fact, given $\mathbf{x} \in M$, consider isothermal coordinates $\{u, v\}$ in a neighbourhood $U_\mathbf{x}$ of $\mathbf{x}$ and take normal fields ν^1 and ν^2 defined on $U_\mathbf{x}$

such that M is ν^j-umbilical, $j = 1, 2$. Without loss of generality we can take ν^1 as the position vector field on M, and ν^2 as a vector field, ν, tangent to $H^4_+(-1)$. Let ξ be another normal field such that $\{\rho, \xi, \nu\}$ defines a pseudo-orthonormal frame for the normal bundle $NU_{\mathbf{x}}$. Then, given any normal field η, we can write $\eta = k_1\rho + k_2\xi + k_3\nu$, for appropriate smooth functions $k_1, k_2, k_3 \colon U \to \mathbb{R}$. The coefficients of the second fundamental form in the direction of η are given by

$$\begin{aligned} e_\eta &= \langle \partial^2\mathbf{X}/\partial x^2, k_1\rho + k_2\xi + k_2\nu\rangle = k_1 e_\rho + k_2 e_\xi + k_3 e_\nu, \\ f_\eta &= \langle \partial^2\mathbf{X}/\partial x\partial y, k_1\rho + k_2\xi + k_3\nu\rangle = k_1 f_\rho + k_2 f_\xi + k_3 f_\nu, \\ g_\eta &= \langle \partial^2\mathbf{X}/\partial y^2, k_1\rho + k_2\xi + k_3\nu\rangle = k_1 g_\rho + k_2 g_\xi + k_3 g_\nu; \end{aligned}$$

and the equation of the curvature lines in these coordinates becomes ([5], [12])

$$\begin{aligned} h(f_\xi du^2 + (g_\xi - e_\xi)dudv - f_\xi dv^2) + k_1(f_{\nu^1} du^2 + (g_{\nu^1} - e_{\nu^1})dudv - f_{\nu^1} dv^2) \\ + k_2(f_{\nu^2} du^2 + (g_{\nu^2} - e_{\nu^2})dudv - f_{\nu^2} dv^2) = 0. \end{aligned}$$

Since M is ρ- and ν-umbilical, we have that $e_\rho(\mathbf{x}) = g_\rho(\mathbf{x})$, $f_\rho(\mathbf{x}) = 0$ and $e_\nu(\mathbf{x}) = g_\nu(\mathbf{x})$, $f_\nu(\mathbf{x}) = 0$ for all $\mathbf{x} \in M$, and thus $f_\rho du^2 + (g_\rho - e_\rho)dudv - f_\rho dv^2 = 0$ and $f_\nu du^2 + (g_\nu - e_\nu)dudv - f_\nu dv^2 = 0$. Therefore, the principal configuration associated to η is given by $h(f_\xi du^2 + (g_\xi - e_\xi)dudv - f_\xi dv^2) = 0$. So both fields η and ξ have the same principal configurations.

Conversely, if $M \subset H^4_+(-1)$, then M is ρ-umbilical, where ρ is the position field. Take $\mathbf{x} \in M$ and let η_1 and η_2 be two linearly independent normal fields defined in a neighborhood $U_{\mathbf{x}}$ of $\mathbf{x}$ in M, that lie in $T_{\mathbf{x}}H^4_+(-1)$ (i.e., they are pseudo-orthogonal to ρ). Their respective principal configurations are given by the equations $f_{\eta_i} du^2 + (g_{\eta_i} - e_{\eta_i})dudv - f_{\eta_i} dv^2 = 0$, for $i = 1, 2$. Since M admits a unique principal configuration, we must have that $f_{\eta_1} = \lambda f_{\eta_2}$ and $g_{\eta_1} - e_{\eta_1} = \lambda(g_{\eta_2} - e_{\eta_2})$, for some function λ on $U_{\mathbf{x}}$. Taking $\bar{\nu} = \eta_1 - \lambda\eta_2$ we have that $f_{\bar{\nu}} = f_{\eta_1} - \lambda f_{\eta_2} = 0$ and $g_{\bar{\nu}} - e_{\bar{\nu}} = g_{\eta_1} - e_{\eta_1} - \lambda(g_{\eta_2} - e_{\eta_2}) = 0$. Therefore, M is $\bar{\nu}$-umbilical. Since $\bar{\nu}$ and ρ are pseudo-orthogonal, they must be linearly independent of $U_{\mathbf{x}}$.

Finally, suppose that θ_1 and θ_2 are the two distinct horoasymptotic fields globally defined on M, and let $b_i, i = 1, 2$ be the corresponding horobinormal fields, which must be distinct too. By taking appropriate coordinates on M we can see that the direction θ_i is a principal direction for the shape operator S_{b_i} and its corresponding principal curvature vanishes everywhere. Since M has a unique principal configuration, we have that the principal configurations of b_1 and b_2 coincide. Therefore θ_2 (resp. θ_1) must be the principal direction of b_1 (resp. b_2) corresponding to the non-vanishing principal curvature. But this means that θ_1 and θ_2 are everywhere orthogonal, except at the critical points of the principal configurations. ∎

Remark: In the case of a surface M immersed in euclidean 4-space, we have that the orthogonality of horoasymptotic directions is a sufficient condition for total semiumbilicity ([13]). This is due to the fact that if $b_i, i = 1, 2$ are the horobinormal fields on M and $k_i, i = 1, 2$ are the corresponding non-vanishing curvatures, then the normal field $\nu = k_2 b_1 - k_1 b_2$ is umbilical over M. In the euclidean case, this is a sufficient condition for semiumbilicity of M. In the horospherical case, we need to require the existence of some umbilical field over M that is everywhere tangent to $H^4_+(-1)$. To be able to ensure this we must have that $\nu = k_2 b_1 - k_1 b_2$ is not a multiple of the position field ρ over M. It is not clear at all that this is always the case for a surface having everywhere orthogonal horoasymptotic fields. Nevertheless, we can assert that under the orthogonality assumption on the horoasymptotic fields, M is totally semiumbilical provided the normal field $\nu = k_2 b_1 - k_1 b_2$ is not a multiple of ρ.

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